The Spacetime Algebra Approach to Massive Classical Electrodynamics with Magnetic Monopoles

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Abstract

Maxwell's equations with massive photons and magnetic monopoles are formulated using spacetime algebra. It is demonstrated that a single non-homogeneous multi-vectorial equation describes the theory. Two limiting cases are considered and their symmetries highlighted: massless photons with magnetic monopoles and finite photon mass in the absence of monopoles. Finally, it is shown that the EM-duality invariance is a symmetry of the Hamiltonian density (for Minkowskian spacetime) and Lagrangian density (for Euclidean 4-space) that reflects the signature of the respective metric manifold.

1 Introduction

Applications of Geometric Algebra (GA) to Maxwell's theory of electromagnetism are known [1] , [2] , [3] , [4]. In this paper the formalism of spacetime algebra is used as in [5] and [6] to formulate classical electrodynamics with massive photons and magnetic monopoles.

The layout of the paper is as follows. In section 2 a brief introduction to the GA of spacetime is presented. In section 3.1 we describe the classical field theory of our model, that is, electric charges interacting with Dirac monopoles via massive photons or Proca fields, in the context of Lagrangian dynamics. In section 3.2 a formulation of Maxwell's theory of electromagnetism with finite photon mass and magnetic monopoles is presented and we demonstrate how to obtain a single nonhomogeneous multi-vectorial equation describing the theory. In section 4 we consider two limiting cases of the general theory: i) massive electrodynamics in the absence of magnetic monopoles and ii) standard electrodynamics with monopoles. In the former case, the loss of local gauge invariance of the theory is discussed. In the latter, the symmetry of the theory under duality rotations is presented. In 5 we consider the invariances of the Lagrangian and Hamiltonian densities, specifically under duality rotations and local gauge transformations.

2 An Outline of Spacetime Algebra

The basic idea in geometric algebra is that of uniting the inner and outer products into a single product, namely the *geometric product*. This product is associative and has the crucial feature of being invertible. The geometric product between two spacetime vectors a and b is defined by

$$ab = a \cdot b + a \wedge b, \tag{1}$$

where $a \cdot b$ is a scalar (a 0-grade multi-vector), while $a \wedge b = i(a \times b)$ is a bi-vector (a grade-2 multi-vector). The quantity i is the unit *pseudoscalar* which will be defined in (9), it is not the unit imaginary number usually employed in physics. For convenience we recall in brief the basic tools of GA that are utilized in our formulation of massive classical electrodynamics in the presence of magnetic monopoles.

The GA of Minkowski spacetime is called *spacetime algebra*. It is generated by four orthogonal basis vectors $\{\gamma_\mu\}_{\mu=0..3}$ satisfying the relations

$$\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) \equiv \eta_{\mu\nu} = diag(+ - - -);\ \mu,\ \nu = 0..3 \tag{2}$$

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$$\gamma_\mu \wedge \gamma_\nu = \frac{1}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right) \equiv \gamma_{\mu\nu}. \tag{3}$$

We observe that (2) and (3) display the same algebraic relations as Dirac's γ-matrices. Indeed, the Dirac matrices constitute a representation of the spacetime algebra. From (2) it is obvious that

$$\gamma_0^2 = 1,\ \gamma_0 \cdot \gamma_i = 0 \text{ and } \gamma_i \cdot \gamma_j = -\delta_{ij};\ i,\ j = 1..3. \tag{4}$$

A basis for this 16-dimensional spacetime Clifford algebra $\mathfrak{cl}(1,3)$ is given by

$$B = \left\{1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, i\gamma_\mu, i\right\}, \tag{5}$$

whose elements represent scalars, vectors, bi-vectors, tri-vectors and pseudoscalars respectively. A general multi-vector M of the spacetime algebra can be written as

$$M = \sum_{k=0}^{4} \langle M \rangle_k = \alpha + a + B + ib + i\beta, \tag{6}$$

where α and β are real scalars, a and b are real spacetime vectors and B is a bi-vector. A general spacetime vector a can be written as

$$a = a^\mu \gamma_\mu. \tag{7}$$

By choosing γ_0 as the future-pointing timelike unit vector, the γ_0-vector determines a map between spacetime vectors a and the even subalgebra of the spacetime $\mathfrak{cl}^+(1,3)$ algebra via the relation

$$a\gamma_0 = a_0 + \vec{a}, \tag{8}$$

where $a_0 = a \cdot \gamma_0$ and $\vec{a}$=$a \wedge \gamma_0$. Notice that the ordinary three-dimensional vector $\vec{a}$ can be interpreted as a spacetime bi-vector $a \wedge \gamma_0$. The geometric interpretation of these relations is the following: since a vector appears to an observer as a line segment existing for a fixed period of time, it is natural that what an observer perceives as a vector should be represented by a spacetime bi-vector. Spacetime is a space of four dimensions with a Lorentz signature, that is, $tr\left(\eta_{\mu\nu}\right) = -2$. The metric $\eta_{\mu\nu}$ has one positive, three negative and no zero eigenvalues. It is this property of the metric that distinguishes the standard $(3+1)$-dimensional space-time from the 4-dimensional space of $SO(4)$, or the $(2+2)$- dimensional spacetime of $SO(2,2)$. The real Clifford algebra $\mathfrak{cl}(1,3)$ is characterized by its total vector dimension $n = p + q$ $(n = 4)$ and signature $s = p - q$ $(s = -2)$ where p is the number of basis vectors with positive norm $(p = 1)$ and q enumerates the basis vectors with negative norm $(q = 3)$. Many authors, especially general relativists, use a Minkowski spacetime metric $+2$. This involves the algebra $\mathfrak{cl}(3,1)$ where the spacelike vectors have positive norm. The algebras $\mathfrak{cl}(1,3)$ and $\mathfrak{cl}(3,1)$ are not isomorphic. In quantum field theory, for instance, a compelling reason to choose $\mathfrak{cl}(1,3)$ over $\mathfrak{cl}(3,1)$ is motivated by the isomorphism $\mathfrak{cl}(1,3) \simeq \mathfrak{cl}(4)$, whereas $\mathfrak{cl}(3,1) \simeq \mathfrak{cl}(2,2)$. In the GA formalism, the metric structure of the space whose geometric algebra is built, reflects the properties of the unit pseudoscalar of the algebra. Indeed, the existence of a pseudoscalar is equivalent to the existence of a metric. In $\mathfrak{cl}(1,3)$ the highest-grade element, the unit pseudoscalar, is defined as,

$$i \overset{\text{def}}{=} \gamma_0\gamma_1\gamma_2\gamma_3. \tag{9}$$

It represents an oriented unit four-dimensional volume element. The corresponding volume element is said to be right-handed because i can be generated from a right-handed vector basis by the oriented product $\gamma_0\gamma_1\gamma_2\gamma_3$. The volume element i has magnitude $|i| = \langle i^\dagger i \rangle_0^{\frac{1}{2}} = 1$, where $\langle x \rangle_0$ denotes the 0-grade component of the multi-vector x. The dagger $\dagger$ is the *reverse* or the *Hermitian adjoint.* For example, given a multivector $x = \gamma_1\gamma_2$, $x^\dagger$ is obtained by reversing the order of vectors in the product. That is, $x^\dagger = \gamma_2\gamma_1 = -\gamma_1\gamma_2$. It is commonly said that i defines an orientation for spacetime. The pseudoscalar satisfies $i^2 = \pm 1$ with the sign depending on the dimension and the signature of the space whose GA is considered. For instance, in spaces of positive definite metric, the pseudoscalar has magnitude $|i| = 1$ while the value of i^2 depends only on the dimension of space as $i^2 = (-1)^{n(n-1)/2}$. This implies that in the positive-definite space of $SO(4)$ and in the zero signature space-time of $SO(2,2)$, the unit pseudoscalar squares to $+1$. For the space-time of the Lorentz group, the pseudoscalar satisfies $i^2 = -1$.

Since we are dealing with a space of even dimension $(n = 4)$, i anticommutes with odd-grade multi-vectors and commutes with even-grade elements of the algebra,

$$i\mathcal{P} = \pm \mathcal{P} i \tag{10}$$

where the multi-vector $\mathcal{P}$ is even for $(+)$ and odd for $(-)$.

An important spacetime vector that will be used in our formulation is the spacetime vector derivative ∇, defined by

$$\nabla = \gamma^{\mu}\partial_{\mu} \equiv \gamma^{0}c^{-1}\partial_{t} + \gamma^{i}\partial_{i}. \tag{11}$$

By post-multiplying with γ^0, we see that

$$\nabla\gamma_0 = c^{-1}\partial_t + \gamma^i\gamma_0\partial_i = c^{-1}\partial_t - \vec{\nabla}, \tag{12}$$

where $\vec{\nabla}$ is the usual vector derivative defined in vector algebra. Similarly, multiplying the spacetime vector derivative by γ^0, we get

$$\gamma_0\nabla = c^{-1}\partial_t + \vec{\nabla}. \tag{13}$$

Finally, we notice that the spacetime vector derivative satisfies the following relation:

$$\Box = (\gamma_0\nabla)(\nabla\gamma_0) = c^{-2}\partial_t^2 - \vec{\nabla}^2, \tag{14}$$

which is the d'Alembert operator used in the description of lightlike traveling waves.

3 EM interaction with electric charges and a magnetic monopole via Proca fields : general case

3.1 Tensor Algebra Formalism

In this section we consider a system of electric charges interacting with magnetic monopoles where the interaction is mediated by massive photons or Proca fields. In cgs units, the Lagrangian density describing such a Maxwell-Proca (MP) system is given by

$$\mathcal{L}_{MP}(A) = -\frac{1}{16\pi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{m_\gamma^2}{8\pi}A_\mu A^\mu - \frac{1}{c}J_\mu A^\mu, \tag{15}$$

where $m_\gamma = \frac{\omega}{c}$ is the inverse of the Compton length associated with the photon mass of the field A_μ, the current J_μ is the electron 4-current, and the 4-vector potential A_μ is associated with the magnetic charge of the monopole. The current decomposes as $J_\mu \equiv \left(\rho,\ -\vec{j}\right)$. The field strength is defined as $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + G_{\mu\nu}$ where the electromagnetic field strength has the usual form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the monopole contribution $G_{\mu\nu}$ is given by

$$G^{\mu\nu}(x) = \frac{4\pi e_m}{c}\int d\tau dx^\mu \delta^{(4)}(x - x_{monopole})\, u^\nu(x), \tag{16}$$

where τ is a timelike parameter and $u^\nu(x) = \frac{dx^\nu}{d\tau}$ denotes the velocity of the monopole. From the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}_{MP}}{\partial A_\mu} - \partial_\nu\left(\frac{\partial\mathcal{L}_{MP}}{\partial(\partial_\nu A_\mu)}\right) = 0, \tag{17}$$

we obtain the field equation

$$\partial_\mu F^{\mu\nu} + m_\gamma^2 A^\nu = \frac{4\pi}{c}J^\nu. \tag{18}$$

By use of the Lorenz gauge condition

$$\partial_\mu A^\mu = 0, \tag{19}$$

(18) can be expressed explicitly in terms of the vector potential,

$$\left(\Box + m_\gamma^2\right)A_\mu = \frac{4\pi}{c}J_\mu. \tag{20}$$

Computing the Bianchi identity $\partial_\mu\, {}^*F^{\mu\nu}$ where the Hodge dual is defined by ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, we obtain

$$\partial_\mu\, {}^*F^{\mu\nu} = -\partial_\mu\, G^{\mu\nu} = -\frac{4\pi}{c}J^\nu_{(m)}, \tag{21}$$

where $J^{\nu}_{(m)} \equiv \left(\rho_m, \vec{j}_{(m)}\right)$, $\rho_m = e_m \delta^3 \left(\vec{x} - \vec{x}_{monopole}\right)$ and $\vec{j}_{(m)} = 0$ since we are in the monopole rest frame.

By decomposing the field equation (18) into its boost and spatial components and using (21), we arrive at the generalized Maxwell equations in presence of massive photons and magnetic monopoles in the vector algebra formalism:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e - m_\gamma^2 A_0, \tag{22}$$

$$\vec{\nabla} \times \vec{E} = -c^{-1}\partial_t \vec{B}, \tag{23}$$

$$\vec{\nabla} \cdot \vec{B} = 4\pi\rho_m, \tag{24}$$

$$\vec{\nabla} \times \vec{B} = 4\pi c^{-1}\vec{j}_e + c^{-1}\partial_t \vec{E} - m_\gamma^2 \vec{A}. \tag{25}$$

Note that although the quantity $-4\pi c^{-1}\vec{j}_m$ is missing in (23) (as it should since we are working in the rest frame of the monopole, $\vec{j}_m = 0$) we will include this term in the following considerations for purposes of generality.

3.2 Spacetime Algebra Formalism

We now recast the generalized vectorial equations (22)-(25) (after including $-4\pi c^{-1}\vec{j}_m$ in (23)) into the language of GA. We begin by utilizing the wedge product (outer product) used in defining the geometric product (1). By means of this operation, (23) and (25) become

$$\vec{\nabla} \wedge \vec{E} = -c^{-1}\partial_t(i\vec{B}) - 4\pi c^{-1} i\vec{j}_m, \tag{26}$$

$$\vec{\nabla} \wedge \vec{B} = i(4\pi c^{-1}\vec{j}_e + c^{-1}\partial_t \vec{E} - m_\gamma^2 \vec{A}). \tag{27}$$

Adding the 0-grade (22) and the 2-grade (26) multi-vectorial equations in which the electric field appears, and computing the equivalent quantity for the magnetic field, we obtain

$$\vec{\nabla}\vec{E} = 4\pi\rho_e - m_\gamma^2 A_0 - c^{-1}\partial_t(i\vec{B}) - 4\pi c^{-1} i\vec{j}_m, \tag{28}$$

$$\vec{\nabla}(i\vec{B}) = -4\pi c^{-1}\vec{j}_e - c^{-1}\partial_t \vec{E} + m_\gamma^2 \vec{A} + 4\pi\rho_m i, \tag{29}$$

where in the last equation we used the fact that the pseudoscalar commutes with the three dimensional vector derivative. Manipulating the two equations in (28) and (29), we obtain

$$(\vec{\nabla} + c^{-1}\partial_t)F = 4\pi c^{-1}(c\rho_e - \vec{j}_e) + 4\pi c^{-1} i(c\rho_m - \vec{j}_m) - m_\gamma^2 A_0 + m_\gamma^2 \vec{A}, \tag{30}$$

where $F \stackrel{\text{def}}{=} \vec{E} + i\vec{B} = E^i\gamma_i\gamma_0 - B^1\gamma_2\gamma_3 - B^2\gamma_3\gamma_1 - B^3\gamma_1\gamma_2$ is the Faraday spacetime bi-vector. Furthermore, defining the spacetime electric and magnetic current j_e and j_m as

$$j_e \stackrel{\text{def}}{=} \left\{ \begin{array}{c} j_e \cdot \gamma_0 = c\rho_e \\ j_e \wedge \gamma_0 = \vec{j}_e \end{array} \right\}, \; j_m \stackrel{\text{def}}{=} \left\{ \begin{array}{c} j_m \cdot \gamma_0 = c\rho_m \\ j_m \wedge \gamma_0 = \vec{j}_m \end{array} \right\} \tag{31}$$

and by using equation (30) and the relations $\gamma_0 i = -i\gamma_0$, $i^2 = -1$ and $\gamma_0^2 = 1$ we arrive at

$$\nabla F = 4\pi c^{-1}(j_e - ij_m) - m_\gamma^2(\gamma_0 A_0 - \gamma_0 \vec{A}). \tag{32}$$

Upon introduction of the spacetime vector potential A

$$A \stackrel{\text{def}}{=} \left\{ \begin{array}{c} A \cdot \gamma_0 = A_0 \\ A \wedge \gamma_0 = \vec{A} \end{array} \right\}, \tag{33}$$

(32) takes the final form

$$\nabla F = 4\pi c^{-1}(j_e - ij_m) - m_\gamma^2 A. \tag{34}$$

This equation represents the GA formulation of the fundamental equations of massive classical electrodynamics in presence of magnetic monopoles. This equation can be decomposed into its vectorial and tri-vectorial components, namely

$$\nabla \cdot F = 4\pi c^{-1} j_e - m_\gamma^2 A, \tag{35}$$

$$\nabla \wedge F = -4\pi c^{-1} i j_m. \tag{36}$$

From equation (34), we determine

$$\begin{aligned}\nabla^2 F &= 4\pi c^{-1}\nabla\cdot j_e + 4\pi c^{-1}\nabla\wedge j_e + 4\pi c^{-1} i\nabla\cdot j_m \\ &\quad +4\pi c^{-1} i\nabla\wedge j_m - m_\gamma^2\nabla\cdot A - m_\gamma^2\nabla\wedge A.\end{aligned} \tag{37}$$

By splitting this equation in its different multi-vectorial parts, we have

$$\nabla^2 F = 4\pi c^{-1}(\nabla\wedge j_e + i\nabla\wedge j_m) - m_\gamma^2\nabla\wedge A, \tag{38}$$

$$4\pi c^{-1}\nabla\cdot j_e - m_\gamma^2\nabla\cdot A = 0, \tag{39}$$

$$4\pi c^{-1} i\nabla\cdot j_m = 0. \tag{40}$$

Equation (39) implies

$$\nabla\cdot j_e = \frac{c}{4\pi} m_\gamma^2 \nabla\cdot A, \tag{41}$$

while from (40) we get

$$i\nabla\cdot j_m = 0. \tag{42}$$

In order to maintain charge conservation $\nabla\cdot j_e = 0$, since $m_\gamma \neq 0$, the Lorenz gauge condition $\nabla\cdot A = 0$ must be satisfied.

4 EM interaction with electric and magnetic charges via Proca fields: special cases

We consider two limiting cases of the GA algebra formulation of massive classical electrodynamics in presence of magnetic monopoles: massive electrodynamics in absence of monopoles and alternatively, standard Maxwell electrodynamics with monopoles.

4.1 Electrodynamics with massive photons

Let's consider the limiting case of equation (34) when no monopoles are present, that is

$$\nabla F = 4\pi c^{-1} j_e - m_\gamma^2 A. \tag{43}$$

Considering the tri-vector and vector parts of (43), we get

$$\nabla\wedge F = 0, \tag{44}$$

$$\nabla\cdot F = 4\pi c^{-1} j_e - m_\gamma^2 A. \tag{45}$$

In the tensor algebra formalism, equation (44) becomes the tensorial relation $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$. Moreover, equation (44) implies that the Faraday bi-vector can be written as an outer derivative of a certain dynamical vector variable A which couples to the external spacetime electric current j_e,

$$F = \frac{1}{2} F^{\mu\nu}\gamma_\mu\wedge\gamma_\nu = \nabla\wedge A \tag{46}$$

where $F^{\mu\nu} = \gamma^\mu\wedge\gamma^\nu\cdot F$ are the components of F in the $\{\gamma^\mu\}$ frame. Substituting this expression of F in equation (45), we get the following equation for A,

$$\nabla^2 A - \nabla(\nabla\cdot A) = 4\pi c^{-1} j_e - m_\gamma^2 A. \tag{47}$$

It is clear that this equation is not invariant under the standard local gauge transformation $A \to A + \nabla\chi(x)$, which implies a loss of local gauge invariance in massive classical electrodynamics. This point will be discussed further in section 5.1.

4.2 Electrodynamics with magnetic charges

Let us consider now the limiting case of equation (34), when the electromagnetic interaction is mediated by massless photons and magnetic monopoles are allowed to occur in the theory. This situation is reflected in the relation

$$\nabla F = 4\pi c^{-1}(j_e - i j_m). \tag{48}$$

Equation (48) is equivalent to the following pair of equations

$$\nabla \cdot F = 4\pi c^{-1} j_e, \tag{49}$$

$$\nabla \wedge F = -4\pi c^{-1} i j_m. \tag{50}$$

This last equation implies the vectorial fields $\overrightarrow{E}$ and $\overrightarrow{B}$ can no longer be described in terms of the dynamical quantity A. Furthermore, equation (48) leads to

$$\nabla^2 F = 4\pi c^{-1}\nabla j_e - 4\pi c^{-1}\nabla(i j_m) = 4\pi c^{-1}\nabla j_e + 4\pi c^{-1} i \nabla j_m \tag{51}$$

that is,

$$\nabla^2 F = 4\pi c^{-1}\nabla \cdot j_e + 4\pi c^{-1}\nabla \wedge j_e + 4\pi c^{-1} i\nabla \cdot j_m + 4\pi c^{-1} i \nabla \wedge j_m. \tag{52}$$

By splitting the different multi-vectorial parts of (52) we find

$$\nabla^2 F = 4\pi c^{-1}\nabla \wedge j_e + 4\pi c^{-1} i \nabla \wedge j_m, \tag{53}$$

$$\nabla \cdot j_e = 0, \tag{54}$$

$$i\nabla \cdot j_m = 0. \tag{55}$$

It is clear from equations (54) and (55) that charge conservation is satisfied.

5 Invariances of the EM Lagrangian and Hamiltonian densities

We now consider the spacetime *duality rotations* (DR) and local gauge invariances of the Lagrangian $\mathcal{L}$ and Hamiltonian $\mathcal{H}$ densities. It is demonstrated that $\mathcal{L}$ (as well as $\mathcal{H}$) looses its local gauge invariance due to the presence of the Proca mass term. Furthermore, $\mathcal{L}$ is shown to be non-invariant under DR. The latter result is well known [7]. In the conventional approach, this reflects the fact that the Lagrangian is an indefinite quantity, that is, there is a relative sign difference between the kinetic and potential energy terms. Since $\left(\vec{E}, \vec{B}\right) \overset{\text{DR}}{\to} \left(-\vec{B}, \vec{E}\right)$ for a rotation of $\alpha = \frac{\pi}{2}$, the Faraday bi-vector transforms as $F \overset{\text{DR}}{\to} -iF$ and one would expect a change in the relative sign of $\mathcal{L}$. Hence, in the free-field electromagnetic case, $\mathcal{L}$ would transform as $\pm\mathcal{L} \overset{\text{DR}}{\to} \mathcal{L}' = \mp\mathcal{L}$. In contrast, since $\mathcal{H}$ is positive definite, $F \overset{\text{DR}}{\to} -iF$ so that $\mathcal{H}'$ is also positive definite, where $\mathcal{H} \overset{\text{DR}}{\to} \mathcal{H}' = \mathcal{H}$.

This being the case, it is interesting to consider the following. The transformation properties of $\mathcal{L}$ and $\mathcal{H}$ under DR is consistent with the fact that in the GA of spacetime, the unit 4-volume element i squares to -1. It thus appears plausible to view the DR invariances of $\mathcal{L}$ and $\mathcal{H}$ as a manifestation of spacetime topology. This point will be further discussed in 5.2. Finally, in accordance with the above arguments $\mathcal{H}$ is shown to be invariant under duality rotations.

5.1 Gauge Invariance

Consider the Maxwell-Proca (MP) Lagrangian density (15) expressed in the GA formalism

$$\mathcal{L}_{MP}^{(GA)}(A) = \alpha_1 \langle F \cdot F \rangle_0 + \alpha_2 \langle A \cdot J \rangle_0 + \alpha_3 m_\gamma^2 \langle A \cdot A \rangle_0 , \tag{56}$$

where α_i are real constants whose values are not specified in our discussion. The Faraday spacetime bi-vector is given by $F = \nabla \wedge A$. The second term of $\mathcal{L}_{MP}^{(GA)}(A)$ describes the coupling between A and the external current J. The last term represents the mass term. Let us discuss the invariance of $\mathcal{L}_{MP}^{(GA)}(A)$ under Local Gauge Transformation (LGT) defined by

$$A \overset{\text{LGT}}{\longrightarrow} A' = A + \nabla\chi(x), \tag{57}$$

where $\chi(x)$ is the scalar gauge field. For the first term we find

$$F \stackrel{\text{LGT}}{\longrightarrow} F' = \nabla \wedge A' = \nabla \wedge (A + \nabla\chi(x)) = \nabla \wedge A + \nabla \wedge (\nabla\chi(x)) = F. \tag{58}$$

The first term of $\mathcal{L}_{MP}^{(GA)}(A)$ is gauge invariant since

$$\langle F \cdot F \rangle_0 \stackrel{\text{LGT}}{\longrightarrow} \langle F' \cdot F' \rangle_0 = \langle F \cdot F \rangle_0 , \tag{59}$$

while for the second term,

$$A \cdot J \to A' \cdot J = A \cdot J - \chi \nabla \cdot J + \nabla \cdot (\chi J) . \tag{60}$$

Since $\nabla \cdot (\chi J)$ is a total divergence it can be ignored because its integral over a finite volume results in a boundary term which can be set to zero. Finally, the gauge invariance of this term is ensured by requiring that the external current J is conserved, $\nabla \cdot J = 0$. We now turn to the last term in $\mathcal{L}_{MP}^{(GA)}(A)$. Under local gauge transformation,

$$A \cdot A \stackrel{\text{LGT}}{\longrightarrow} A' \cdot A' = A^2 + (\nabla\chi)^2 + 2\nabla \cdot (\chi A) - 2\chi \nabla \cdot A. \tag{61}$$

Using the Lorentz condition $\nabla \cdot A = 0$ and ignoring $\nabla \cdot (\chi A)$ for reasons described above, we obtain,

$$A \cdot A \stackrel{\text{LGT}}{\longrightarrow} A' \cdot A' = A^2 + (\nabla\chi)^2 . \tag{62}$$

Clearly, this term is not gauge invariant due to the occurrence of the non-vanishing square gradient $(\nabla\chi)^2$. Local gauge symmetry is therefore lost in electromagnetic interactions mediated by massive photons.

5.2 EM Duality and 4D spacetime signature

Two fundamental properties characterize the physical space: its signature and its dimensionality $D = 4$ [8]. An explanation of why spacetime has $3 + 1$ signature rather than $4 + 0$ or $2 + 2$ metric is discussed in [9].

It is shown in [10] that there is a relation between four signs in electrodynamics and the signature: i) the "$-$" sign in the Ampere-Maxwell law, $\vec{\nabla} \times \vec{B} - c^{-1}\partial_t \vec{E} = 0$; ii) the "$+$" sign in the Faraday law (Lenz rule), $\vec{\nabla} \times \vec{E} + c^{-1}\partial_t \vec{B} = 0$; iii) the "$+$" sign in the electromagnetic density, $u_{EM} = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right)$; iv) the $(+,-,-,-)$ signature of the Lorentz metric. In [10] it is also shown that given a manifold endowed with a certain signature, the electric and magnetic energy densities are positive in the Minkowskian case, while they have opposite sign in the Euclidean electrodynamics [11, 12]. These results will be briefly discussed in the GA formalism and, in addition, it will be shown that the EM duality invariance of the Lagrangian and Hamiltonian densities in the Minkowskian and Euclidean cases are related to the signature of the metric of the manifold over which the dynamics is constructed.

5.2.1 The Minkowskian case

For the sake of simplicity, assume $c = 1$. Consider the Lenz law with "$+$" sign,

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{63}$$

We will show that the "$+$" sign will lead to the correct relativistic wave equation. Considering the curl of (63) and introducing the outer product "$\wedge$", we obtain

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = i_{\mathbf{M}}^2 \nabla \cdot \left(\vec{\nabla} \wedge \vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) \tag{64}$$

where the subscript $\mathbf{M}$ represents Minkowski spacetime. In absence of sources, $\vec{\nabla} \cdot \vec{E} = 0$, and recalling that $\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0$, we obtain

$$\Box_{\mathbf{M}} \vec{E} \equiv \left(i_{\mathbf{M}}^2 \vec{\nabla}^2 + \frac{\partial^2}{\partial t^2}\right) \vec{E} = 0 \tag{65}$$

where $i_{\mathbf{M}} = \gamma_0\gamma_1\gamma_2\gamma_3$ is the Minkowski spacetime unit pseudoscalar that satisfies $i_{\mathbf{M}}^2 = -1$. The GA formalism emphasizes the fact that the shape of the second order differential operator $\Box_{\mathbf{M}}$ reflects the

signature of the metric. Moreover, in the GA formalism the Lagrangian density for electromagnetism in the absence of sources is proportional to

$$^{\mathbf{M}}\mathcal{L}_{\gamma}^{(GA)} \propto \left\langle F^2 \right\rangle_0 = \vec{E}^2 - \vec{B}^2, \tag{66}$$

where,

$$F^2 = \left\langle F^2 \right\rangle_0 + \left\langle F^2 \right\rangle_4 = \vec{E}^2 - \vec{B}^2 + 2i_{\mathbf{M}}\vec{E}\cdot\vec{B}. \tag{67}$$

Notice that $\left\langle F^2 \right\rangle_0$ and $\left\langle F^2 \right\rangle_4$ are Lorentz invariant terms. Under Minkowski Duality Rotations (MDR) with an arbitrary angle α, the *Faraday spacetime bi-vector* $F = \vec{E} + i_{\mathbf{M}}\vec{B}$ transforms as

$$F \longrightarrow F' = Fe^{-i_{\mathbf{M}}\alpha}. \tag{68}$$

For the special case $\alpha = \frac{\pi}{2}$, $(\vec{E},\vec{B}) \overset{\text{MDR}}{\to} (\vec{B},-\vec{E})$, F transforms as

$$F \overset{\text{MDR}}{\longrightarrow} F' = -i_{\mathbf{M}}F. \tag{69}$$

Thus

$$\left\langle F^2 \right\rangle_0 \overset{\text{MDR}}{\longrightarrow} \left\langle F'^2 \right\rangle_0 = i_{\mathbf{M}}^2 \left\langle F^2 \right\rangle_0 = -\left\langle F^2 \right\rangle_0, \tag{70}$$

where we used the fact that the spacetime unit volume $i_{\mathbf{M}}$ (the unit pseudoscalar) squares to -1. Finally, from (62) and (66) we conclude that under MDR $^{\mathbf{M}}\mathcal{L}_{\gamma}^{(GA)}$ changes its sign as,

$$\mathcal{L}_{\gamma}^{(GA)} \overset{\text{MDR}}{\longrightarrow} -\mathcal{L}_{\gamma}^{(GA)}. \tag{71}$$

The significant point here is that *electromagnetic duality* is a *symmetry* that is not exhibited as an *invariance* of the *Lagrangian density* and moreover, in the GA formalism this reflects the signature of spacetime; or more specifically the fact that the the unit pseudoscalar squares to -1. It can be easily shown that the free-field electromagnetic energy-momentum tensor $T(a) = -\frac{1}{2}FaF$, where $a = a^{\mu}\gamma_{\mu}$, is either gauge-invariant or invariant under MDR. Furthermore, notice that $\left\langle F^2 \right\rangle_4$ is not invariant under MDR.

On the other hand, the Hamiltonian density is proportional to

$$^{\mathbf{M}}\mathcal{H}_{\gamma}^{(GA)} \propto \left\langle FF^{\dagger} \right\rangle_0 = \vec{E}^2 + \vec{B}^2, \tag{72}$$

where

$$FF^{\dagger} = \left\langle FF^{\dagger} \right\rangle_0 + \left\langle FF^{\dagger} \right\rangle_2 = \vec{E}^2 + \vec{B}^2 - 2i_{\mathbf{M}}\vec{E}\wedge\vec{B} \tag{73}$$

and the dagger † is the *reverse* or the *Hermitian adjoint*. For example, given a multivector $A = \gamma_1\gamma_2$, $A^{\dagger}$ is obtained by reversing the order of vectors in the product. That is, $A^{\dagger} = \gamma_2\gamma_1 = -\gamma_1\gamma_2$. Notice that $\left\langle FF^{\dagger} \right\rangle_0$ is not Lorentz invariant. Under MDR with $\alpha = \frac{\pi}{2}$, we have

$$\left\langle FF^{\dagger} \right\rangle_0 \overset{\text{MDR}}{\longrightarrow} \left\langle F'F'^{\dagger} \right\rangle_0 = -i_{\mathbf{M}}^2 \left\langle FF^{\dagger} \right\rangle_0 = \left\langle FF^{\dagger} \right\rangle_0. \tag{74}$$

Therefore, from (68) and (70) the Hamiltonian density $^{\mathbf{M}}\mathcal{H}_{\gamma}^{(GA)}$ is invariant under electromagnetic duality rotation:

$$^{\mathbf{M}}\mathcal{H}_{\gamma}^{(GA)} \overset{\text{MDR}}{\longrightarrow} {}^{\mathbf{M}}\mathcal{H}_{\gamma}^{(GA)}. \tag{75}$$

Notice that $\left\langle FF^{\dagger} \right\rangle_2 \equiv \frac{8\pi}{c}\vec{S}$, where $\vec{S}$ is the Poynting vector, is invariant under MDR also. In view of the meaning of the oriented 4-volume element i, and considering the transformation properties of the Hamiltonian density under MDR, we are lead to conclude that, *the EM-duality invariance of the free-field electromagnetic Hamiltonian density reflects the signature of the spacetime whose GA is built.* The duality invariance of $^{\mathbf{M}}\mathcal{H}_{\gamma}^{(GA)}$ is fundamental in order to preserve the positive definiteness of energy density.

Phase and spacetime duality rotations are transformations that occur on different spaces: the former correspond to internal gauge rotations in the canonical phase space while the latter are external (spacetime) Lorentz transformations (LT). We note however that this MDR ($\alpha = \pi/2$) can be obtained as special cases of LT. By requiring the quantities $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}'$ and $\vec{E}^2 - \vec{B}^2 = \vec{E}'^2 - \vec{B}'^2$ be invariant, it follows that $\left(\vec{E},\vec{B}\right) \overset{\text{LT}}{\to} \left(\vec{E}',\vec{B}'\right)$ such that $E_iB_j\delta_{ij} = 0$ and $E_iE_j\delta_{ij} = B_iB_j\delta_{ij}$ mimics the desired DT. This suggests a link between spacetime and gauge symmetries. Indeed, in the canonical formulation, a duality phase rotation

emerges in ${}^{\mathbf{M}}\mathcal{L} \to {}^{\mathbf{M}}\mathcal{L}' = e^{-2i_{\mathbb{C}}\phi}\ {}^{\mathbf{M}}\mathcal{L}$. Considering the GA equivalent ${}^{\mathbf{M}}\mathcal{L}^{(GA)} \overset{\alpha=\pi/2}{\longrightarrow} {}^{\mathbf{M}}\mathcal{L}'^{(GA)} = e^{-2i_{\mathbf{M}}\alpha}$ ${}^{\mathbf{M}}\mathcal{L}^{(GA)}$, we arrive at the *correspondence* $e^{-2i_{\mathbb{C}}\phi} \sim i_{\mathbf{M}}^2$. This link between spacetime signature $i_{\mathbf{M}}^2$ and gauge phase factor $e^{-2i_{\mathbb{C}}\phi}$ at this stage is purely mathematical. The geometrization program of physics is not sufficiently advanced to provide an adequate interpretation of gauge symmetry. An understanding of any potential link between these two types of invariances requires a deeper geometrical analysis of local gauge invariance. For this purpose, the GA formalism seems to be most adequately suited.

5.2.2 The Euclidean case

The Euclidean Maxwell equations in absence of sources in flat space are [12],

$$\vec{\nabla} \cdot \vec{E} = 0,\ \vec{\nabla} \cdot \vec{B} = 0,\ \vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0,\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0. \tag{76}$$

The remarkable difference from the usual vacuum Maxwell equations is that there is an "anti-Lenz" law, with "+" sign replaced by a "−" sign. A main consequence of such a change is that in Euclidean 4-spaces there is no propagation with finite speed,

$$\Box_{\mathbf{E}}\vec{E} \equiv \left(i_{\mathbf{E}}^2\vec{\nabla}^2 + \frac{\partial^2}{\partial t^2}\right)\vec{E} = 0, \tag{77}$$

where the subscript $\mathbf{E}$ represents Euclidean 4-space, $i_{\mathbf{E}} = e_0e_1e_2e_3$ is the 4-space Euclidean unit pseudoscalar satisfying $i_{\mathbf{E}}^2 = 1$. The multivectors e_j satisfy

$$e_0^2 = 1,\ e_0 \cdot e_i = 0 \text{ and } e_i \cdot e_j = \delta_{ij};\ i,\ j = 0..3. \tag{78}$$

The monochromatic wave solutions of (77) are either exponentially growing or decaying as a function of distance along the direction of propagation. It is clear, using the GA formalism that the unit pseudoscalar of the algebra encodes information about the structure of the differential operator describing wave propagation in the spaces whose GA is constructed. Finally, notice that the Euclidean Maxwell equations in (76) are invariant under the Euclidean Duality Rotation (EDR) $(\vec{E},\vec{B}) \overset{\text{EDR}}{\to} (\vec{B},\vec{E})$. In terms of the Euclidean Faraday bi-vector $F = \vec{E} + i_{\mathbf{E}}\vec{B}$, the Lagrangian density becomes

$${}^{\mathbf{E}}\mathcal{L}_{\gamma}^{(GA)} \propto \left\langle F^2 \right\rangle_0 = \vec{E}^2 + \vec{B}^2, \tag{79}$$

where $F^2 = \vec{E}^2 + \vec{B}^2 + 2i_{\mathbf{E}}\vec{E}\cdot\vec{B}$. Also, the Hamiltonian density is,

$${}^{\mathbf{E}}\mathcal{H}_{\gamma}^{(GA)} \propto \left\langle FF^{\dagger} \right\rangle_0 = \vec{E}^2 - \vec{B}^2, \tag{80}$$

where $FF^{\dagger} = \vec{E}^2 - \vec{B}^2 - 2i_{\mathbf{E}}\vec{E}\wedge\vec{B}$. The GA formalism emphasizes the fact that the definiteness of the Hamiltonian and Lagrangian densities are related to the signature of the metric manifold and that only the positive-definite quantities preserve the duality invariance.

6 Conclusions

Spacetime algebra is used to unify Maxwell's equations with non-vanishing photon mass and Dirac monopoles. The theory is described by a non-homogeneous multi-vectorial equation. The physical content of the theory is not obscured by reference to specific choices of frames or set of coordinates. Moreover, gauge and spacetime invariances of the theory have been considered in the GA formalism. In addition to reproducing all standard invariances, it is shown that the GA approach also supports the idea of an EM origin for the Lorentzian signature of spacetime. It is shown that the EM duality invariance of the Lagrangian and Hamiltonian densities in the Minkowskian and Euclidean cases are related to the signature of the metric of the manifold over which the dynamics is constructed. Moreover, GA formalism makes clear that DR are symmetries of the definite quantities in the Minkowskian and Euclidean manifolds, namely Hamiltonian and Lagrangian densities respectively.

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